Distribution-Free Approach for Stochastic Joint-Replenishment Problem with Backorders-Lost Sales Mixtures, and Controllable Major Ordering Cost and Lead Times

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Abstract
In this paper, we study the periodic-review Joint-Replenishment Problem (JRP) with stochastic demands and backorders-lost sales mixtures. We assume that lead times are made of two major components: a common part to all items and an item-specific portion. We further suppose that the item-specific component of lead times and the major ordering cost are controllable. To reflect the practical circumstance characterized by the lack of complete information about the demand distribution, we adopt the minimax distribution-free approach. That is, we assume that only the mean and the variance of the demand can be evaluated. The objective is to determine the strict cyclic replenishment policy, the length of (the item-specific component of) lead times, and the major ordering cost that minimize the long-run expected total cost. To approach this minimization problem, we present a first optimization algorithm. However, numerical tests highlighted how computationally expensive this algorithm would be for a practical application. Therefore, we then propose two alternative heuristics. Extensive numerical experiments have been carried out to investigate the performance of the developed algorithms. Results have shown that the proposed alternative heuristics are actually efficient and seem therefore promising for a practical application.

Keywords: Inventory; Joint-Replenishment Problem; Stochastic; Optimization; Heuristics; Distribution-free procedure.

1. Introduction

In practice, there exist conditions under which coordinated replenishments may be economically beneficial. Goyal [1] cited cases where the output of a batch production is placed in different packages, or where different items are all procured from the same supplier. Questions related to the optimization of such systems belong to the Joint-Replenishment Problem (JRP).

In the standard JRP, each item purchased from the same supplier is characterized by an ordering cost that is charged every time that item is ordered and is independent of the amount of units ordered. Moreover, there exists a major ordering cost that is incurred for each order, independently of the variety of items procured. This cost structure reveals economies of scale that can be exploited with the combination of different items in the same order (Khouja and Goyal [2]; Kiesmüller [3]).

In literature, numerous works about the JRP can be found. The most recent review considers papers published between the years 1989 and 2005 (Khouja and Goyal [2]). More recently, additional papers have been presented, though. We can classify papers into two main groups, depending on whether demands are deterministic or stochastic: deterministic JRP and stochastic JRP. In the deterministic JRP group, Narayanan and Robinson [4] presented two heuristics to solve the capacitated, dynamic lot-sizing JRP with time-varying demands. Tsao and Sheen [5] studied a two-actor, multi-item supply chain with a credit period and weight freight cost discounts. Zhang *et al.* [6] developed a JRP model with complete backordering where demands of some minor items are correlated with that of a major item. Amaya *et al.* [7] presented a new heuristic approach based on linear programming to solve the JRP under deterministic demands and resource constraints. Tsao and Teng [8] developed two heuristics to solve the JRP under deterministic demands and trade credit. Wang *et al.* [9–11] used meta-heuristics and fuzzy set-based modelling to approach the problem. In the stochastic JRP group, Paul *et al.* [12] studied a JRP model in presence of a random percentage of defective units in each replenishment and with price discount, considering deterministic and constant demands. Kiesmüller [3] compared two different continuous-review policies, assuming that the demand process is a compound renewal process, and taking into account a constraint on the total amount of products to be ordered. Narayanan and Robinson [13] carried out a study to evaluate the performances of nine joint replenishment lot-sizing heuristics and policy design variables in a dynamic rolling schedule environment with normally distributed demands. Tanrikulu *et al.* [14] developed a continuous-review policy taking into account a constraint on the transportation capacity. They assumed that demand for each item follows an independent Poisson process.

Real inventory systems are typically subject to various uncertainties; therefore, a model that include random aspects is more practical. In a stochastic environment, an important issue is to reduce or, rather, to control replenishment lead times. According to the Just-in-Time (JIT) philosophy, reduced lead times may lower the safety stock, improve the customer service level, and reduce both the stockout loss and the expected total costs (Glock [15]). Although controllable lead time is an important aspect in inventory management, it has rarely been considered in the JRP.

Another key aspect of JIT is ordering/setup cost reduction, which may lead to improved quality and flexibility, stock reduction, and increased effective capacity. Although several researchers have taken into account this issue (*e.g.*, Chuang *et al.* [16]; Ouyang *et al.* [17]; Lin [18]; Sarkar and Moon [19]; Lou and Wang [20]), little effort has been made to include it into the JRP framework.

Stochastic inventory models should not neglect backorders-lost sales mixtures. In fact, it is reasonable to assume that only a fraction of the demand during the stockout period is backordered, while the remaining quota is lost. For example, customers whose needs are not critical can wait (these demands are backordered); while others cannot wait and require their needs be satisfied elsewhere (these demands are lost). Numerous inventory models include this feature (see, *e.g.*, Vijayan and Kumaran [21]; Chang and Lo [22]; Sicilia *et al.* [23]; Wang and Tang [24]). However, in the JRP, it has seldom been addressed.

In some practical situations, information about the demand distribution may be rather limited. That is, the decision-maker may only know an estimate of the mean and of the variance, but not the specific distribution type. In this circumstance, the traditional approach is to treat the demand within a given period as a normally distributed random variable (Moon and Gallego [25]). This also follows from the assumption that individual demands are independent and identically distributed (i.i.d.) random variables, and then, according to the central limit theorem, the gaussianity of their sum can readily be deduced. However, this procedure is hardly valid in reality, as single demands are generally not i.i.d. random variables (Andersson *et al.* [26]). In addition, one should also consider that the normal distribution is not recommended for items characterized by demand with a large coefficient of variation (Gallego *et al*. [27]).

Under this condition, *i.e*., (*i*) when only an estimate of the mean and of the variance of the demand is available and (*ii*) when it is not possible/practical to hypothesize a specific demand distribution, it is reasonable to follow a conservative procedure (Moon and Gallego [25]). That is, the replenishment policy is optimized considering the worst non-negative distribution with the given mean and variance. This is called "minimax distribution-free approach". Due to its practicality (easy to use) and optimality (under certain conditions), it has received great attention in the inventory management literature. The reader is referred to some of the most recent works, *e.g*., Sarkar *et al*. [28]; Kumar and Goswami [29]; Sarkar and Mahapatra [30]; and Raza [31]. In this paper, we include this approach into the JRP context. To the best of our knowledge, this is the first attempt to apply the distribution-free procedure to the JRP.

Owing to what said above, this paper investigates the periodic-review stochastic JRP with backorders-lost sales mixtures under the minimax distribution-free approach. We assume that lead times are made of two major components: a common part to all items and an item-specific portion. We further suppose that the item-specific component of lead times and the major ordering cost are controllable. The purpose is to determine the strict cyclic replenishment policy, the length of (the item-specific component of) lead times, and the major ordering cost that minimize the long-run expected total cost.

We present a first optimization algorithm, which may however result computationally expensive for a practical application, as numerical tests showed. Hence, to overcome this limitation, we then propose two efficient alternative heuristics. Although they follow the same logic, they differ in the fact that one of them works on an approximated expression of the expected total cost function obtained by means of an *ad hoc* Taylor series expansion. Numerical experiments will initially serve to investigate the performance of the developed algorithms. Then, we numerically analyse the error that our distribution-free model achieves with respect to the case in which the demand in the protection interval is Gaussian.

The remainder is as follows. Section 2 defines the notation, the assumptions, and the problem. Section 3 and Section 4 present the first optimization algorithm and the alternative heuristics, respectively. Section 5 deals with the numerical study. Section 6 discusses conclusions and further remarks.

2. Notation, assumptions and problem definition

We set the stage by describing the inventory system under exam. We consider a family of items procured from one supplier. The inventory of each item is managed according to a periodic-review policy, taking into consideration a stochastic demand. Each item is characterized by a minor ordering cost paid every time the item is purchased, which happens at regular time intervals specified by an integer multiple of a basic cycle time. The minor ordering cost of a given item is independent of the other products. A major ordering cost is charged with frequency established by the basic cycle time. This cost is independent of the number of items acquired.

Each item features a deterministic lead time made of two main parts: a common component to all products and an item-specific component.

We adopt the following notation:

Decision variables

T	Cycle time or review period, *i.e.*, time interval between orders (time unit).
L_n	Item-specific component of the lead time of item *n* (time unit).
z_n	Safety factor of item *n*.
R_n	Target level of item *n*. An equivalent decision variable to z_n (quantity unit).
k_n	Integer multiplier of the replenishment cycle time *T* relevant to item *n*.
A	Major ordering cost (money/order).

Parameters

a_n	Minor ordering cost of item *n* (money/order).
h_n	Unit holding cost of item *n* (money/quantity unit/time unit).
ρ_n	Fixed penalty cost per unit shortage of item *n* (money/quantity unit).
π_n	Marginal profit per unit of item *n* (money/quantity unit).
D_n	Average demand rate of item *n* (quantity units/time unit).
σ_n	Standard deviation of demand rate of item *n* (quantity unit/time unit).
β_n	Fraction of shortage (*i.e.*, demand during the stockout period) of item *n* that will be lost.
L	Common lead-time component to all items (time unit).

Random variables

X_n	Demand of item *n* during its protection interval (*i.e.*, $k_nT + L_n + L$).

Functions and operators

$f_n(\cdot)$	Probability density function (p.d.f.) of X_n.
$E[\cdot]$	Mathematical expectation.
x^+	Maximum between 0 and *x*, *i.e.*, $x^+ \equiv \max\{0, x\}$.
$\lfloor x \rfloor$	Smallest integer greater than or equal to *x*.
$\mathbf{1}_A(\cdot)$	Indicator function on the set *A*.

Sets

$\mathcal{R}$	Real numbers.
$\mathcal{N}$	Natural numbers.

Classes

$\mathcal{F}_n$	Class of p.d.f.s with finite mean $\bar{\mu}_n \equiv D_n\left(k_nT + L_n + L\right)$ and standard deviation $\bar{\sigma}_n \equiv \sigma_n\sqrt{k_nT + L_n + L}$.

We adopt the following main assumptions:

- *N* items are considered, *i.e.*, $n = 1, 2, ..., N$, which are ordered from the same supplier.
- For each *n*, $f_n \in \mathcal{F}_n$.
- The demands are mutually independent.
- The lead time l_n of item *n* is given by $l_n \equiv L + L_n$. The component *L* is deterministic and constant; while L_n is deterministic but controllable according to the formulation given below.
- At least one item is ordered every *T* time units.
- Inventory of item *n* is reviewed every k_nT time units. A sufficient quantity is ordered up to the target level R_n and the ordering quantity arrives after l_n time units.

- For each *n*, there is no more than a single order outstanding, *i.e.*, $l_n \le k_n T$.
- For each *n*, the distribution of X_n is unknown/unspecified and only its mean and variance can be evaluated.
- For each *n*, the target inventory level R_n is given by $R_n = \bar{\mu}_n + z_n \bar{\sigma}_n$, where $\bar{\mu}_n \equiv D_n \left(k_n T + l_n \right)$ and $\bar{\sigma}_n \equiv \sigma_n \sqrt{k_n T + l_n}$.
- For each *n*, shortages are allowed and partially backordered with ratio $1-\beta_n$. The fraction of shortage with ratio β_n is lost.

We assume that the item-specific component L_n of the *n*th item lead time is controllable with a similar crashing cost formulation to that adopted by, *e.g*., Chuang *et al*. [16] or Lin [18]. That is, we assume that L_n is made of M_n mutually independent, deterministic and constant components. The generic m*th* component has a minimum duration $b_{m,n}$, a normal duration $s_{m,n}$ and a crashing cost per time unit $c_{m,n}$, with $c_{1,n} \le c_{2,n} \le ... \le c_{M_n,n}$. Such components are crashed one at a time starting with the component of least $c_{m,n}$ and so on. If $L_{m,n}$ is the length of L_n with components $1,2,...,m$ crashed to their minimum duration, then we have:

$$L_{m,n} \equiv L_{0,n} - \left[\left(s_{1,n} - b_{1,n} \right) + \left(s_{2,n} - b_{2,n} \right) + ... + \left(s_{m,n} - b_{m,n} \right) \right],$$

where $L_{0,n} \equiv \sum_{m=1}^{M_m} s_{m,n}$. The crashing cost $U_n \left(L_n \right)$ relevant to L_n is thus expressed as follows:

$$U_n \left(L_n \right) \equiv \sum_{m=1}^{M_n} \mathbf{1}_{\left\{ L_n : L_{m,n} \le L_n \le L_{m-1,n} \right\}} \left(L_n \right) U_{m,n} \left(L_n \right), \ L_n \in \left[L_{M_n,n}, L_{0,n} \right], \tag{1}$$

where

$$U_{m,n} \left(L_n \right) \equiv c_{m,n} \left(L_{m-1,n} - L_n \right) + \left[c_{1,n} \left(s_{1,n} - b_{1,n} \right) + c_{2,n} \left(s_{2,n} - b_{2,n} \right) + ... + c_{m-1,n} \left(s_{m-1,n} - b_{m-1,n} \right) \right].$$

We can observe that $U_n \left(L_n \right)$ is a piecewise-linear, decreasing function in the interval $\left[L_{M_n,n}, L_{0,n} \right]$, where it is also continuous and convex.

Differently from standard inventory models where the major ordering cost *A* is considered a parameter, we assume that *A* is controllable by means of a capital investment $I\left(A\right)$. This investment is required to reduce the ordering cost from the original level A_0 to a target level *A*, with $0 < A \le A_0$. For example, $I\left(A\right)$ may be regarded as an investment of purchasing a more efficient vehicle, or an investment of new technology to facilitate the transport.

The function $I\left(A\right)$ is the one-time investment cost whose benefits will extend to the long-term into the future. Hence, if τ is the annual fractional cost of capital investment (*e.g*., interest rate), then $\tau I\left(A\right)$ is the annual cost of such an investment. We assume that $I\left(A\right)$ follows a logarithmic investment function:

$$I\left(A\right) \equiv \frac{1}{\delta} \ln\left(\frac{A_0}{A} \right), \text{ with } 0 < A \le A_0, \tag{2}$$

where δ is the percentage decrease in *A* per money unit increase in investment. We can note that $I\left(A\right)$ is a convex, strictly decreasing function. Many researchers have adopted this logarithmic investment function for ordering (or setup) cost reduction. We can cite, for example, Chuang *et al.* [16]; Ouyang *et al.* [17]; Lin [18]; and Sarkar and Moon [19].

Under our assumptions, the expected total cost per time unit relevant to the *n*th item is

$$\bar{C}_n\left(T,k_n,R_n,L_n\right)=\frac{a_n}{k_nT}+h_n\left(R_n-D_nl_n-\frac{D_nk_nT}{2}+\beta_nE\left[\left(X_n-R_n\right)^+\right]\right) \\ +\frac{\bar{\pi}_n}{k_nT}E\left[\left(X_n-R_n\right)^+\right]+\frac{U_n\left(L_n\right)}{k_nT}, \tag{3}$$

where $\bar{\pi}_n\equiv\rho_n+\pi_n\beta_n$. It consists of the ordering cost, the inventory holding cost, the shortage cost, and the lead time crashing cost. This formulation readily follows from that of a single and independent item (*i.e.*, with $N=1$), and noting that the review period of the *n*th item, in the JRP, is k_nT. Details about the derivation of the cost function in the single-item case can be found, *e.g.*, in Annadurai and Uthayakumar [32].

If we let $\mathbf{k}\equiv\left(k_1,k_2,...,k_N\right)$, $\mathbf{z}\equiv\left(z_1,z_2,...,z_N\right)$, and $\mathbf{L}\equiv\left(L_1,L_2,...,L_N\right)$, then the long-run expected total cost per time unit for a family of *N* items, taking also into account the cost of capital investment to reduce the major ordering cost, is

$$\bar{C}\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)=\tau I\left(A\right)+\frac{A}{T}+\sum_{n=1}^{N}\bar{C}_n\left(T,k_n,R_n,L_n\right). \tag{4}$$

Our optimization problem can be formulated as follows:

$$\min_{(T,A,\mathbf{k},\mathbf{z},\mathbf{L})} \quad \bar{C}\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right), \tag{5}$$

$$\text{s.t.} \quad 0<A\leq A_0, \tag{6}$$

$$k_n=1 \text{ for some } n, \tag{7}$$

$$L_n\in\left[L_{M_n,n},L_{0,n}\right] \text{ for each } n, \tag{8}$$

$$T>0, \tag{9}$$

$$\mathbf{z}\in\mathcal{R}^N, \tag{10}$$

$$\mathbf{k}\in\mathcal{N}^N. \tag{11}$$

Note that constraint (6) characterizes the so-called strict cyclic policy.

We recall that we are in the case where, for each *n*, the distribution of X_n is unknown/unspecified and only its mean and variance can be evaluated. In this circumstance, problem (5) under constraints (6–11) cannot be solved directly. In fact, since we do not know f_n, for each *n*, the quantity $E\left[\left(X_n-R_n\right)^+\right]$ cannot explicitly be calculated.

To overcome this issue, we follow the minimax distribution-free procedure. The minimax principle consists in choosing each f_n as the most unfavourable p.d.f. in $\mathcal{F}_n$ for each $\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)$, and then minimizing over $\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)$. Although this is a conservative approach, there are several supporting arguments that can be raised. First, it can easily be applied in practice: statistical tables or computer programs that work with distribution functions are not required. It is worth noting that this also permits to obtain analytically tractable expressions. Secondly, it is optimal under some conditions (Moon and Gallego [25]; Gallego and Moon [33]). Finally, we would observe that, under a mathematical viewpoint, a two-point demand distribution is often assumed (Fleischhacker and Fok [34]).

In place of problem (5), we therefore consider the following problem:

$$\min_{(T,A,\mathbf{k},\mathbf{z},\mathbf{L})}\max_{\{f_n|f_n\in\mathcal{F}_n\}}\bar{C}\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right). \tag{12}$$

Since (*i*) the maximum is determined over $\left\{f_n\mid f_n\in\mathcal{F}_n\right\}$ for a fixed vector $\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)$, and (*ii*) the functions $\bar{C}_n\left(T,k_n,R_n,L_n\right)$, for $n=1,2,...,N$, are mutually independent (random variables X_n are mutually independent), then (12) can be rewritten as follows:

$$\min_{(T,A,\mathbf{k},\mathbf{z},\mathbf{L})}\left[\tau I(A)+\frac{A}{T}+\sum_{n=1}^{N}\max_{f_n\in\mathcal{F}_n}\bar{C}_n\left(T,k_n,R_n,L_n\right)\right]. \tag{13}$$

Problem (13) can be simplified with the following proposition:

Proposition 1. *For each n,*

$$E\left[\left(X_n-R_n\right)^+\right]\le\frac{1}{2}\left\{\sqrt{\sigma_n^2\left(k_nT+l_n\right)+\left[R_n-D_n\left(k_nT+l_n\right)\right]^2}-\left[R_n-D_n\left(k_nT+l_n\right)\right]\right\}. \tag{14}$$

Inequality (14) is valid for any $f_n\in\mathcal{F}_n$. *Moreover, the upper bound (14) is tight.*

Proposition 1 derives from Chuang *et al.* [16], once recalled that the review period of the *n*th item is k_nT. Further details can be found, for example, in Moon and Gallego [25] and Gallego *et al.* [27].

According to Proposition 1 and to the definition of R_n, (13) is finally reduced to

$$\min_{(T,A,\mathbf{k},\mathbf{z},\mathbf{L})}C\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right), \tag{15}$$

with

$$C\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)=\tau I(A)+\frac{A}{T}+\sum_{n=1}^{N}C_n\left(T,k_n,z_n,L_n\right), \tag{16}$$

where

$$\begin{aligned}C_n\left(T,k_n,z_n,L_n\right)=&\frac{a_n}{k_nT}+\\&+h_n\left(\frac{k_nTD_n}{2}+z_n\sigma_n\sqrt{k_nT+l_n}+\frac{1}{2}\beta_n\sigma_n\sqrt{k_nT+l_n}\left(\sqrt{1+z_n^2}-z_n\right)\right)+\\&+\frac{\bar{\pi}_n}{2k_nT}\sigma_n\sqrt{k_nT+l_n}\left(\sqrt{1+z_n^2}-z_n\right)+\frac{U_n\left(L_n\right)}{k_nT}.\end{aligned} \tag{17}$$

Ultimately, instead of solving problem (5) directly (that is not possible as the distribution of X_n is unknown/unspecified), we turn to minimizing $C\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)$ under constraints (6–11), whose solution evidently gives an upper bound to the minimum cost of the original problem.

3. First optimization algorithm

The first algorithm to minimize $C\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)$ under constraints (6–11) exploits the following result:

Proposition 2. $C\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)$ *satisfies the following properties:*

i. *For* $\left(T,A,\mathbf{k},\mathbf{z}\right)$ *fixed and* $L_n\in\left[L_{m,n},L_{m-1,n}\right]$ *for each n with* $m=0,1,...,M_n$, $C\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)$ *is strictly concave in* $\mathbf{L}$.

ii. *For* $\left(T,\mathbf{k},\mathbf{z},\mathbf{L}\right)$ *fixed,* $C\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)$ *is strictly convex in A. Moreover, the First Order Condition in A gives:*

$$A=\bar{A}(T)\equiv\frac{\tau}{\delta}T. \tag{18}$$

iii. *For* $\left(\mathbf{k},\mathbf{L}\right)$ *fixed and* $A=\bar{A}(T)$, $C\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)$ *is convex in* $\left(T,\mathbf{z}\right)$. *Moreover, for values of T such that* $\bar{\pi}_n>h_nk_nT\left(1-\beta_n\right)$, *the First Order Condition in* z_n *gives:*

$$z_n=\bar{z}_n\left(k_nT\right)\equiv\frac{\bar{\pi}_n-h_nk_nT\left(2-\beta_n\right)}{2\sqrt{h_nk_nT\left[\bar{\pi}_n-h_nk_nT\left(1-\beta_n\right)\right]}}\text{, for each } n. \tag{19}$$

iv. For $(T, A, \mathbf{z}, \mathbf{L})$ *fixed, and relaxing the integer constraint on* $\mathbf{k}$, $C(T, A, \mathbf{k}, \mathbf{z}, \mathbf{L})$ *is convex in* $\mathbf{k}$. *Moreover,* $C(T, A, \mathbf{k}, \mathbf{z}, \mathbf{L}) \to +\infty$ *as* k_n *tends to* 0^+ *or* $+\infty$, *for each n.*

Proof. Properties at points (*i*) and (*ii*) are quite straightforward to prove. The convexity properties given at points (*iii*) and (*iv*) can readily be deduced observing that $C_n(T, k_n, z_n, L_n)$ is convex in (T, z_n) (as proved by Annadurai and Uthayakumar [32], considering item *n* as independent) and that convexity is invariant under affine maps. Finally, the limit property at point (*iv*) is relatively easy to observe.

□

From property (*i*) we have that, for $(T, A, \mathbf{k}, \mathbf{z})$ fixed and $L_n \in [L_{m,n}, L_{m-1,n}]$ for each *n*, the minimum of $C(T, A, \mathbf{k}, \mathbf{z}, \mathbf{L})$ in $\mathbf{L}$ is a vector $\bar{\mathbf{L}}$ whose generic *n*th component lies on one of the endpoints of $[L_{m,n}, L_{m-1,n}]$, with $m = 0,1,\dots,M_n$. That is, we are allowed to write:

$$\min_{(T,A,\mathbf{k},\mathbf{z},\mathbf{L})} C(T, A, \mathbf{k}, \mathbf{z}, \mathbf{L}) = \min\Big\{ \min_{(T,A,\mathbf{k},\mathbf{z})} C(T, A, \mathbf{k}, \mathbf{z}, \bar{\mathbf{L}}) \,|\, \bar{\mathbf{L}} \equiv (L_{m,1}, L_{m,2}, \dots, L_{m,N}) \text{ with } m = 0,1,\dots,M_n \ \forall\ n = 1,2,\dots,N \Big\}. \tag{20}$$

Practically, the minimization problem within the braces has to be solved for each vector $\bar{\mathbf{L}}$. It is worth noting that the total number of such vectors is $\prod_n (M_n + 1)$.

Proposition 2 allows us to derive a simplified expression of $C(T, A, \mathbf{k}, \mathbf{z}, \mathbf{L})$ that makes easier the optimization process. First, it is possible to observe that

$$\sqrt{1+z_n^2} = z_n \frac{\beta_n h_n k_n T + \bar{\pi}_n}{\bar{\pi}_n - h_n k_n T(2-\beta_n)}, \quad n = 1,2,\dots,N, \tag{21}$$

which follows from the First-Order Condition of optimality with respect to z_n. If we now put the expression of $\sqrt{1+z_n^2}$ into $C_n(T, k_n, z_n, L_n)$ and impose the First-Order Condition with respect to z_n, with some algebraic manipulations (16) becomes:

$$C(T, A, \mathbf{k}, \mathbf{L}) = \tau I(A) + \frac{A}{T} + \sum_{n=1}^{N} C_n(T, k_n, L_n), \tag{22}$$

where

$$C_n(T, k_n, L_n) = \frac{a_n + U_n(L_n)}{k_n T} + h_n \frac{k_n T D_n}{2} + \sigma_n \sqrt{h_n} \sqrt{\frac{(k_n T + l_n)(\bar{\pi}_n - h_n k_n T(1-\beta_n))}{k_n T}}, \quad n = 1,2,\dots,N. \tag{23}$$

Equation (20) can conveniently be rewritten with respect to $C(T, A, \mathbf{k}, \mathbf{L})$:

$$\min_{(T,A,\mathbf{k},\mathbf{L})} C(T, A, \mathbf{k}, \mathbf{L}) = \min\Big\{ \min_{(T,A,\mathbf{k})} C(T, A, \mathbf{k}, \bar{\mathbf{L}}) \,|\, \bar{\mathbf{L}} \equiv (L_{m,1}, L_{m,2}, \dots, L_{m,N}) \text{ with } m = 0,1,\dots,M_n \ \forall\ n = 1,2,\dots,N \Big\}. \tag{24}$$

Ultimately, the following algorithm, which is based on the above results, gives the solution $(T^*, A^*, \mathbf{k}^*, \mathbf{z}^*, \mathbf{L}^*)$ and the corresponding cost C^* to the problem of minimizing $C(T, A, \mathbf{k}, \mathbf{z}, \mathbf{L})$ under constraints (6–11):

Algorithm 1.

Step 1. Set $\mathcal{C} \leftarrow \{\varnothing\}$.

Step 2. For each vector $\bar{\mathbf{L}} \equiv \left(L_{m,1}, L_{m,2}, ..., L_{m,N}\right)$, with $m = 0,1,...,M_n$ and $n = 1,2,...,N$, do Steps 2.1-2.3.

Step 2.1. Set $i = 1$, $\hat{C}_1 = 0$, $\hat{C}_0 = +\infty$, $\mathbf{L} \leftarrow \bar{\mathbf{L}}$.

Step 2.2. While $\hat{C}_i \neq \hat{C}_{i-1}$, do Steps 2.2.1-2.2.4.

Step 2.2.1. Let $\mathcal{K}$ be the set of all integer vectors $\mathbf{k}$ with $0 < k_n \leq i$, for each *n*, and $k_n = 1$ for some *n*. Moreover, let $\mathcal{C}_i \equiv \{\varnothing\}$.

Step 2.2.2. For each vector $\mathbf{k}$ in $\mathcal{K}$, do Steps 2.2.2.1-2.2.2.2.

Step 2.2.2.1. With $A = \bar{A}(T)$, set $\tilde{T}_i \leftarrow \arg\min_T C(T, A, \mathbf{k}, \mathbf{L})$.

Step 2.2.2.2. If $0 < \bar{A}(\tilde{T}_i) \leq A_0$, then set $\tilde{A}_i \leftarrow \bar{A}(\tilde{T}_i)$ and go to Step 2.2.2.4. Otherwise, set $\tilde{A}_i \leftarrow A_0$ and go to Step 2.2.2.3.

Step 2.2.2.3. With $A = \tilde{A}_i$, set $\tilde{T}_i \leftarrow \arg\min_T C(T, A, \mathbf{k}, \mathbf{L})$.

Step 2.2.2.4. Update $\mathcal{C}_i$: $\mathcal{C}_i \leftarrow \mathcal{C}_i \cup \left\{C\left(\tilde{T}_i, \tilde{A}_i, \mathbf{k}, \mathbf{L}\right)\right\}$.

Step 2.2.3. Set $\hat{C}_i \leftarrow \min \mathcal{C}_i$ and $\left({}^*T_i, {}^*A_i, {}^*\mathbf{k}_i, \mathbf{L}\right) \leftarrow \arg\min \mathcal{C}_i$.

Step 2.2.4. Set $i \leftarrow i + 1$.

Step 2.3. Update $\mathcal{C}$: $\mathcal{C} \leftarrow \mathcal{C} \cup \left\{\hat{C}_i\right\}$.

Step 3. Set $\left(T^*, A^*, \mathbf{k}^*, \mathbf{L}^*\right) \leftarrow \arg\min \mathcal{C}$, $z_n^* \leftarrow \bar{z}_n\left(k_n^* T^*\right)$ for $n = 1,2,...,N$, and $C^* \leftarrow C\left(T^*, A^*, \mathbf{k}^*, \mathbf{L}^*\right)$.

Numerical experiments have shown that Algorithm 1 is actually effective but computationally onerous. The minimization problem at Steps 2.2.2.1 and 2.2.2.3 can be approached, *e.g.*, with a standard numerical method or with a meta-heuristic algorithm. In fact, the First-Order Condition of optimality in *T* cannot be solved in closed form.

We recall that, even under deterministic conditions, the JRP is highly complex: it is NP-hard (Arkin *et al.* [35]) and determining the optimal policy may be computationally prohibitive for large problems (Khouja and Goyal [2]). The development of a more efficient solution procedure is therefore strongly encouraged to enhance the applicability in practice. This is evidently even truer for the problem under consideration.

4. Two alternative solution methods

Owing to the observations raised at the end of the previous section, in the next two sections we give two alternative heuristics to approach the problem of minimizing $C(T, A, \mathbf{k}, \mathbf{z}, \mathbf{L})$ under constraints (6–11).

4.1. First heuristic

This heuristic is adapted from the improved version of the original Silver's algorithm (Kaspi and Rosenblatt [36]). Its derivation procedure is described below.

Let us put $\mathbf{L} = \bar{\mathbf{L}}$ fixed. For each *n*, let ${}^*T_n$ be the minimum of $C_n\left(T, k_n, L_{m,n}\right)$ in $T_n \equiv k_n T$, which can be found with a standard numerical method. The item with the smallest ${}^*T_n$ needs to be replenished most often; therefore, its multiplier will have the smallest possible value, *i.e.*, 1. If we denote such item with index $n = 1$, then we have $k_1 = 1$, and $C\left(T, A, \mathbf{k}, \bar{\mathbf{L}}\right)$ can be rewritten as follows:

$$
\begin{aligned}
C\left(T,A,\mathbf{k},\bar{\mathbf{L}}\right)= \quad & \tau I\left(A\right)+\frac{A}{T} \\
& +\frac{a_1+U_1\left(L_{m,1}\right)}{T}+h_1\frac{TD_1}{2} \\
& +\sigma_1\sqrt{h_1}\sqrt{\frac{\left(T+L_{m,1}+L\right)\left(\bar{\pi}_1-h_1T\left(1-\beta_1\right)\right)}{T}} \\
& +\sum_{n=2}^{N}\left[\frac{a_n+U_n\left(L_{m,n}\right)}{k_nT}+h_n\frac{k_nTD_n}{2}\right. \\
& \left.+\sigma_n\sqrt{h_n}\sqrt{\frac{\left(k_nT+L_{m,n}+L\right)\left(\bar{\pi}_n-h_nk_nT\left(1-\beta_n\right)\right)}{k_nT}}\right].
\end{aligned}
\tag{25}
$$

If we replace A with $\bar{A}\left(T\right)$ in (25), this becomes:

$$
\begin{aligned}
C\left(T,\mathbf{k},\bar{\mathbf{L}}\right)= \quad & \xi\ln\left(\frac{A_0}{\xi T}\right)+\xi \\
& +\frac{a_1+U_1\left(L_{m,1}\right)}{T}+h_1\frac{TD_1}{2}+\sigma_1\sqrt{h_1}\sqrt{\frac{\left(T+L_{m,1}+L\right)\left(\bar{\pi}_1-h_1T\left(1-\beta_1\right)\right)}{T}} \\
& +\sum_{n=2}^{N}\left[\frac{a_n+U_n\left(L_{m,n}\right)}{k_nT}+h_n\frac{k_nTD_n}{2}\right. \\
& \left.+\sigma_n\sqrt{h_n}\sqrt{\frac{\left(k_nT+L_{m,n}+L\right)\left(\bar{\pi}_n-h_nk_nT\left(1-\beta_n\right)\right)}{k_nT}}\right].
\end{aligned}
\tag{26}
$$

Let us relax the integrality constraint on k_n, with $n=2,3,...,N$. If we first take the partial derivatives of $C\left(T,\mathbf{k},\bar{\mathbf{L}}\right)$ with respect to T and k_n, with $n=2,3,...,N$, and then impose the First-Order Conditions, we obtain:

$$
\begin{aligned}
\frac{\partial}{\partial T}C\left(T,\mathbf{k},\bar{\mathbf{L}}\right)= \quad & -\frac{\xi}{T}-\frac{a_1+U_1\left(L_{m,1}\right)}{T^2}+\frac{D_1h_1}{2}-\frac{\sigma_1\left(1-\beta_1\right)\sqrt{Th_1^3}}{2\sqrt{\left(T+L_{m,1}+L\right)\left(\bar{\pi}_1-h_1T\left(1-\beta_1\right)\right)}} \\
& -\frac{\bar{\pi}_1\sigma_1\sqrt{h_1}\left(L_{m,1}+L\right)}{2\sqrt{T^3\left(T+L_{m,1}+L\right)\left(\bar{\pi}_1-h_1T\left(1-\beta_1\right)\right)}} \\
& +\sum_{n=2}^{N}\left[-\frac{a_n+U_n\left(L_{m,n}\right)}{k_nT^2}+\frac{D_nh_nk_n}{2}\right. \\
& -\frac{\sigma_n\left(1-\beta_n\right)\sqrt{Th_n^3k_n^3}}{2\sqrt{\left(k_nT+L_{m,n}+L\right)\left(\bar{\pi}_n-h_nk_nT\left(1-\beta_n\right)\right)}} \\
& \left.-\frac{\bar{\pi}_n\sigma_n\sqrt{h_n}\left(L_{m,n}+L\right)}{2\sqrt{k_nT^3\left(k_nT+L_{m,n}+L\right)\left(\bar{\pi}_1-h_nk_nT\left(1-\beta_n\right)\right)}}\right]=0,
\end{aligned}
\tag{27}
$$

$$\frac{\partial}{\partial k_n}C\left(T,\mathbf{k},\bar{\mathbf{L}}\right)= -\frac{a_1+U_1\left(L_{m,1}\right)}{k_n^2 T}+\frac{D_n h_n T}{2}-\frac{\sigma_1\left(1-\beta_1\right)\sqrt{k_n T^3 h_n^3}}{2\sqrt{\left(k_n T+L_{m,1}+L\right)\left(\bar{\pi}_n-h_n k_n T\left(1-\beta_n\right)\right)}} -\frac{\bar{\pi}_n\sigma_n\sqrt{h_n}\left(L_{m,n}+L\right)}{2\sqrt{k_n^3 T\left(k_n T+L_{m,n}+L\right)\left(\bar{\pi}_n-h_n k_n T\left(1-\beta_n\right)\right)}}=0,\ \forall\ n=2,\dots,N. \tag{28}$$

Multiplying (28) by k_n/T and substituting into (27), we get:

$$Z\left(T\right)\equiv -\frac{\xi}{T}-\frac{a_1+U_1\left(L_{m,1}\right)}{T^2}+\frac{D_1 h_1}{2}-\frac{\sigma_1\left(1-\beta_1\right)\sqrt{T h_1^3}}{2\sqrt{\left(T+L_{m,1}+L\right)\left(\bar{\pi}_1-h_1 T\left(1-\beta_1\right)\right)}} -\frac{\bar{\pi}_1\sigma_1\sqrt{h_1}\left(L_{m,1}+L\right)}{2\sqrt{T^3\left(T+L_{m,1}+L\right)\left(\bar{\pi}_1-h_1 T\left(1-\beta_1\right)\right)}}=0. \tag{29}$$

It is possible to observe that $Z\left(T\right)$ is the derivative with respect to T of $C_1\left(T,k_1,L_{m,1}\right)$, with $k_1=1$, plus the term $-\xi/T$. Due to Proposition 2, we can deduce that the equation $Z\left(T\right)=0$ admits a unique solution in *T*, denoted with $\tilde{T}$.

If we substitute *T* with $\tilde{T}$ in (28) and multiply it by $1/\tilde{T}$, we can see that

$\left[\frac{\partial}{\partial k_n}C\left(T,\mathbf{k},\bar{\mathbf{L}}\right)\right]\cdot\frac{1}{\tilde{T}}=0 \Leftrightarrow k_n\tilde{T}={}^{*}T_n$, for each $n=2,\dots,N$.

Since the values ${}^{*}T_n$ have been calculated previously, we can put $k_n={}^{*}T_n/\tilde{T}$. Exploiting the unimodality of $C_n\left(T,k_n,L_{m,n}\right)$ in $T_n\equiv k_n T$ and letting $q_n\equiv\lfloor k_n\rfloor$, we evidently choose to use q_n instead of q_n+1 if and only if $C_n\left(\tilde{T},q_n,L_{m,n}\right)\le C_n\left(\tilde{T},q_n+1,L_{m,n}\right)$. In this case, we have ${}^{*}k_n\equiv q_n$, otherwise ${}^{*}k_n\equiv q_n+1$. We recall that ${}^{*}k_1=1$.

The near-optimal value ${}^{*}T$ of *T* (for a given $\bar{\mathbf{L}}$ and for $A=\bar{A}\left(T\right)$) has to be evaluated taking into account the (integer) values ${}^{*}k_n$ just obtained. That is, ${}^{*}T$ must be determined by solving (27) with k_n replaced by ${}^{*}k_n$ for each *n*. Note that, thanks to Proposition 2, we can argue that the equation $\partial C\left(T,\mathbf{k},\bar{\mathbf{L}}\right)/\partial T=0$ admits a unique solution in *T*.

Once ${}^{*}T$ is obtained, it is necessary to verify whether $0<\bar{A}\left({}^{*}T\right)\le A_0$ or not. Let ${}^{*}A$ be the near-optimal value of *A* (for a given $\bar{\mathbf{L}}$). If $0<\bar{A}\left({}^{*}T\right)\le A_0$, then we can put ${}^{*}A=\bar{A}\left({}^{*}T\right)$. Otherwise, we put $A=A_0={}^{*}A$ (in virtue of the convexity of $C\left(T,A,\mathbf{k},\bar{\mathbf{L}}\right)$ in *A*), and then we repeat the above procedure working on the function $C\left(T,A,\mathbf{k},\bar{\mathbf{L}}\right)$ to find a new vector $\left({}^{*}T,{}^{*}k_2,\dots,{}^{*}k_N\right)$ (clearly, the item with index $n=1$, *i.e.*, with the smallest ${}^{*}T_n$, does not change). Note that the vector $\left({}^{*}T,{}^{*}A,{}^{*}k_1,\dots,{}^{*}k_N\right)$ is the near-optimal solution in $\left(T,A,\mathbf{k}\right)$, for a given $\bar{\mathbf{L}}$.

According to (24), to obtain a near-optimal solution to the problem of minimizing $C\left(T,A,\mathbf{k},\mathbf{z},\mathbf{L}\right)$ under constraints (6–11), it is necessary to repeat the whole procedure above described over all vectors $\bar{\mathbf{L}}$. In conclusion, the above procedure is summarized in the following algorithm, which gives the solution $\left(T^{*},A^{*},\mathbf{k}^{*},\mathbf{z}^{*},\mathbf{L}^{*}\right)$ and the corresponding cost C^{*}:

Algorithm 2.

Step 1. Set $\mathcal{C}\leftarrow\{\varnothing\}$.

Step 2. For each vector $\bar{\mathbf{L}} \equiv (L_{m,1}, L_{m,2}, ..., L_{m,N})$, with $m = 0,1,...,M_n$ and $n = 1,2,...,N$, do Steps 2.1-2.9.

Step 2.1. For each *n*, determine ${}^{*}T_n$ by minimizing $C_n(T, k_n, L_{m,n})$ in $k_n T$.

Step 2.2. Let $n = 1$ be the index of the item with smallest ${}^{*}T_n$. Set $k_1 \leftarrow {}^{*}k_1 = 1$.

Step 2.3. Calculate $\tilde{T}$ by solving the equation $Z(T) = 0$.

Step 2.4. For each $n = 2,3,...,N$, set $q_n \leftarrow \lfloor {}^{*}T_n / \tilde{T} \rfloor$.

Step 2.5. For each $n = 2,3,...,N$, if $C_n(\tilde{T}, q_n, L_{m,n}) \le C_n(\tilde{T}, q_n + 1, L_{m,n})$, then set ${}^{*}k_n \leftarrow q_n$. Otherwise, set ${}^{*}k_n \leftarrow q_n + 1$.

Step 2.6. Calculate ${}^{*}T$ by solving (27) with k_n replaced by ${}^{*}k_n$ for each *n*.

Step 2.7. If $0 < \bar{A}({}^{*}T) \le A_0$, then set ${}^{*}A \leftarrow \bar{A}({}^{*}T)$ and go to Step 2.8. Otherwise, set ${}^{*}A \leftarrow A_0$ and go to Step 2.7.1.

Step 2.7.1. Set $A \leftarrow A_0$, $\xi = 0$, and $a_1 \leftarrow a_1 + A$.

Step 2.7.2. Do Steps 2.3-2.6.

Step 2.7.3. Reset the actual values of ξ and a_1: $\xi \leftarrow \tau/\delta$ and $a_1 \leftarrow a_1 - A$.

Step 2.8. Update $\mathcal{C}$: $\mathcal{C} \leftarrow \mathcal{C} \cup \{\hat{C}({}^{*}T, {}^{*}A, {}^{*}\mathbf{k}, \bar{\mathbf{L}})\}$, where ${}^{*}\mathbf{k} \equiv ({}^{*}k_1, {}^{*}k_2, ..., {}^{*}k_N)$.

Step 3. Set $(T^*, A^*, \mathbf{k}^*, \mathbf{L}^*) \leftarrow \arg\min \mathcal{C}$, $C^* \leftarrow C(T^*, A^*, \mathbf{k}^*, \mathbf{L}^*)$, and $z_n^* \leftarrow \bar{z}(k_n^* T^*)$ for each *n*.

A remark is needed. If we would first determine the *n*th component L_n^* of $\mathbf{L}^*$ as the value of L_n that minimizes $C_n(T, k_n, L_{m,n})$ in $(k_n T, L_n)$, then this does not assure that the final solution is at least as good as the one obtained with Algorithm 2. In fact, by doing so $\mathbf{L}^*$ is determined with no consideration about the integer constraint on each k_n.

4.2. Second heuristic

The second heuristic exploits an approximation of $C(T, A, \mathbf{k}, \bar{\mathbf{L}})$ obtained by replacing part of $C_n(T, k_n, L_{m,n})$, for each *n*, with an *ad hoc* Taylor series expansion. Eynan and Kropp [37] adopted a similar approximation method.

In particular, we replace $\sqrt{(k_n T + L_{m,n} + L)(\bar{\pi}_n - h_n k_n T(1-\beta_n))(k_n T)^{-1}}$ with its second-order Taylor series expansion in $T_n \equiv k_n T$ in a neighbourhood of $\bar{T}_n \equiv \sqrt{2(a_n + U_n(L_{m,n}))(h_n D_n)^{-1}}$. Note that $\bar{T}_n$ is the optimum cycle time under deterministic conditions and considering *n* as a single and independent item.

With reference to a neighbourhood of $\bar{T}_n$, we can write:

$$\sqrt{\frac{(T_n + L_{m,n} + L)(\bar{\pi}_n - h_n k_n T(1-\beta_n))}{T_n}} \approx p_{0,n} + p_{1,n}(T_n - \bar{T}_n) + \frac{1}{2} p_{2,n}(T_n - \bar{T}_n)^2, \qquad (30)$$

where:

$$p_{0,n} \equiv \sqrt{\frac{(\bar{T}_n + L_{m,n} + L)(\bar{\pi}_n - h_n \bar{T}_n(1-\beta_n))}{\bar{T}_n}},$$

$$p_{1,n} \equiv -\frac{\bar{\pi}_n(L_{m,n} + L) + h_n(1-\beta_n)\bar{T}_n^2}{2\bar{T}_n^{\frac{3}{2}}\sqrt{(\bar{T}_n + L_{m,n} + L)(\bar{\pi}_n - h_n \bar{T}_n(1-\beta_n))}},$$

$$p_{2,n} \equiv \frac{1}{4\bar{T}_n^{\frac{5}{2}}\left[\left(\bar{T}_n + L_{m,n} + L\right)\left(\bar{\pi}_n - h_n\bar{T}_n\left(1-\beta_n\right)\right)\right]^{\frac{3}{2}}}\left[\bar{\pi}_n^2 L_{m,n}\left(3\left(L_{m,n}+L\right)+4\bar{T}\right)\right.$$
$$\left. -2h_n\bar{\pi}_n\left(L_{m,n}+L\right)\bar{T}_n\left(1-\beta_n\right)\left(2\left(L_{m,n}+L\right)+3\bar{T}_n\right)-h_n^2\bar{T}_n^4\left(\beta_n-1\right)^2\right].$$

According to (30) and with some algebraic manipulations, we can approximate $C_n\left(T,k_n,L_{m,n}\right)$ in a neighbourhood of $\bar{T}_n$ as follows:

$$C_n\left(T,k_n,L_{m,n}\right) \approx \hat{C}_n\left(T,k_n,L_{m,n}\right) = \frac{u_n}{k_nT} + v_nk_nT + w_n\left(k_nT\right)^2 + y_n \text{, for each } n=1,2,...,N\text{,} \tag{31}$$

where,

$u_n \equiv a_n + U_n\left(L_{m,n}\right)$,

$v_n \equiv \dfrac{h_nD_n}{2} + \sigma_n\sqrt{h_n}\left(p_{1,n} - p_{2,n}\bar{T}_n\right)$,

$w_n \equiv \dfrac{1}{2}p_{2,n}\sigma_n\sqrt{h_n}$,

$y_n \equiv \sigma_n\sqrt{h_n}\left(p_{0,n} - p_{1,n}\bar{T}_n + \dfrac{1}{2}p_{2,n}\bar{T}_n^2\right)$.

Note that all coefficients are given for a fixed vector $\bar{\mathbf{L}}$, and that $\hat{C}_n\left(T,k_n,L_{m,n}\right)$ is structured as the cost function in deterministic conditions plus a constant and a quadratic term in k_nT.

Finally, taking into account (31), we can write:

$$C\left(T,A,\mathbf{k},\bar{\mathbf{L}}\right) \approx \hat{C}\left(T,A,\mathbf{k},\bar{\mathbf{L}}\right) = \tau I\left(A\right) + \frac{A}{T} + \sum_{n=1}^{N}\left[\frac{u_n}{k_nT} + v_nk_nT + w_n\left(k_nT\right)^2 + y_n\right]. \tag{32}$$

Clearly, $\hat{C}\left(T,A,\mathbf{k},\bar{\mathbf{L}}\right)$ is only an approximation of $C\left(T,A,\mathbf{k},\bar{\mathbf{L}}\right)$. However, numerical experiments have shown that it provides a good estimate of the actual cost in a reasonably wide range of parameter values (see the numerical study section).

The heuristic proposed in this section adopts the same logic as that of Algorithm 2, with the only difference that it works on $\hat{C}\left(T,A,\mathbf{k},\bar{\mathbf{L}}\right)$, instead of $C\left(T,A,\mathbf{k},\bar{\mathbf{L}}\right)$. Since its derivation procedure is basically identical to that of Algorithm 2, the related details can therefore be omitted. If we let $\left(T^*,A^*,\mathbf{k}^*,\mathbf{z}^*,\mathbf{L}^*\right)$ and C^* be a near-optimal solution and the corresponding cost, respectively, the second heuristic works as follows:

Algorithm 3.

Step 1. Set $\mathcal{C} \leftarrow \{\varnothing\}$.

Step 2. For each vector $\bar{\mathbf{L}} \equiv \left(L_{m,1},L_{m,2},...,L_{m,N}\right)$, with $m=0,1,...,M_n$ and $n=1,2,...,N$, do Steps 2.1-2.9.

Step 2.1. For each *n*, let ${}^*T_n$ be the unique positive root of $N_{1,n}\left(T_n\right) \equiv 2w_nT_n^3 + v_nT_n^2 - u_n$.

Step 2.2. Let $n=1$ be the index of the item with smallest ${}^*T_n$, and set $k_1 \leftarrow {}^*k_1 = 1$.

Step 2.3. Let $\tilde{T}$ be the unique positive root of $N_2\left(T\right) \equiv 2w_1T^3 + v_1T^2 - \xi T - u_1$.

Step 2.4. For each $n=2,3,...,N$, set $q_n \leftarrow \left\lfloor {}^*T_n/\tilde{T} \right\rfloor$.

Step 2.5. For each $n=2,3,...,N$, if $\hat{C}_n\left(\tilde{T},q_n,L_{m,n}\right) \le \hat{C}_n\left(\tilde{T},q_n+1,L_{m,n}\right)$, then set ${}^*k_n \leftarrow q_n$. Otherwise, set ${}^*k_n \leftarrow q_n + 1$.

Step 2.6. Let ${}^{*}T$ be the unique positive root of

$$N_3(T) \equiv \left(2\sum_{n=1}^{N} w_n {}^{*}k_n^2\right)T^3 + \left(\sum_{n=1}^{N} v_n {}^{*}k_n\right)T^2 - \xi T - \left(u_1 + \sum_{n=2}^{N} \frac{u_n}{{}^{*}k_n}\right).$$

Step 2.7. If $0 < \bar{A}\left({}^{*}T\right) \le A_0$, then set ${}^{*}A \leftarrow \bar{A}\left({}^{*}T\right)$ and go to Step 2.8. Otherwise, set ${}^{*}A \leftarrow A_0$ and go to Step 2.7.1.

Step 2.7.1. Set $A \leftarrow A_0$, $\xi = 0$, and $u_1 \leftarrow u_1 + A$.

Step 2.7.2. Do Steps 2.3-2.6.

Step 2.7.3. Reset the actual values of ξ and u_1: $\xi \leftarrow \tau/\delta$ and $u_1 \leftarrow u_1 - A$.

Step 2.8. Update $\mathcal{C}$: $\mathcal{C} \leftarrow \mathcal{C} \cup \left\{\hat{C}\left({}^{*}T, {}^{*}A, {}^{*}\mathbf{k}, \bar{\mathbf{L}}\right)\right\}$, where ${}^{*}\mathbf{k} \equiv \left({}^{*}k_1, {}^{*}k_2, ..., {}^{*}k_N\right)$.

Step 3. Set $\left(T^*, A^*, \mathbf{k}^*, \mathbf{L}^*\right) \leftarrow \arg\min \mathcal{C}$, $C^* \leftarrow C\left(T^*, A^*, \mathbf{k}^*, \mathbf{L}^*\right)$, and $z_n^* \leftarrow \bar{z}\left(k_n^* T^*\right)$ for each *n*.

It is worth noting that setting $\xi = 0$ at Step 2.7.1 is equivalent (to our purposes) to making null the derivative of $\tau I(A)$ when $A = A_0$. In fact, in such circumstance, $\tau I(A)$ is simply a constant.

We would observe that ${}^{*}T_n$, $\tilde{T}$, and ${}^{*}T$ are determined under the hypothesis that u_n, v_n, w_n, and y_n are positive real numbers, for each *n*. If fact, ${}^{*}T_n$, $\tilde{T}$, and ${}^{*}T$ are determined as the *unique* positive real roots of specific cubic equations. These roots can be found in closed form according to the procedure proposed by Nickalls [38].

Since $\hat{C}\left(T, A, \mathbf{k}, \bar{\mathbf{L}}\right)$ is very similar to the cost function in deterministic conditions, it is much simpler than $C\left(T, A, \mathbf{k}, \bar{\mathbf{L}}\right)$. Also note that, contrarily to Algorithm 2, Algorithm 3 does not need the use of any iterative minimization technique. We therefore reasonably expect that the computational time required by Algorithm 3 is less than that of Algorithm 2.

5. Numerical study

This section shows the result of extensive numerical experiments aimed first to investigate the performance of the proposed algorithms, and secondly to analyse the error that our distribution-free model achieves with respect to a case where the demand distribution is known. In these tests, the cycle time is expressed in years and the item-specific part of lead times consists of three components. Parameter values range in the intervals shown in Table 1. Note that this table reports the coefficient of variation $Cv_n \equiv \sigma_n / D_n$ rather than the standard deviation σ_n, which is therefore obtained as $\sigma_n = Cv_n \cdot D_n$. The quantities τ and δ take the values 0.1 and 1/5800, respectively (Ouyang *et al.* [17]). The common component L of lead times is supposed to be zero.

Parameters	Ranges
A_0	[100, 220]
a_n	[100, 220]
h_n	[1, 25]
D_n	[100, 1000]
Cv_n	[0.01, 0.40]
β_n	[0.1, 0.9]
π_n	[80, 150]
ρ_n	[20, 70]
$s_{m,n}$	[17, 25]
$b_{m,n}$	[7, 15]

$c_{1,n}$	[0.2, 1.0]
$c_{2,n}$	[1.8, 3.2]
$c_{3,n}$	[4, 6]

Table 1. Intervals where parameters take values.

The experiments have been made on a PC with an Intel® Core™ i7 processor at 2.4GHz and with 16GB of RAM. MATLAB® R2013b has been used as computing environment. Since the nonlinear minimization problems involved in Algorithm 1 and Algorithm 2 are unidimensional, they have been solved with the *fminbnd* function.

The numerical study has also considered a hybrid Genetic Algorithm (GA), which has been adopted as a benchmark algorithm. GA has been implemented to solve the following problem for each fixed vector $\bar{\mathbf{L}}$:

$$\begin{aligned} \min_{\mathbf{k}} \quad & C\left({}^{*}T, {}^{*}A, \mathbf{k}, \bar{\mathbf{L}}\right), \\ \text{s.t.} \quad & k_n = 1 \text{ for some } n, \\ & \mathbf{k} \in \mathcal{N}^N, \end{aligned}$$

where ${}^{*}T$ and ${}^{*}A$ are the respective values of T and A minimizing $C\left(T, A, \mathbf{k}, \bar{\mathbf{L}}\right)$ for a given $\mathbf{k}$. To find ${}^{*}T$ and ${}^{*}A$, we have adopted the interior-point algorithm. The reader will note that GA has been integrated with a constrained nonlinear minimization algorithm; this is the reason why we have used the term "hybrid GA". GA has been set with default parameter values within Optimization Toolbox™, except for the following ones, which have been tuned to assure the convergence in a reasonable time and to improve the solution score:

- Population size: $15 \cdot N$.
- Elite count: $\lfloor 0.2 \cdot 15 \cdot N \rfloor$.
- Crossover fraction: 0.6.
- Migration direction: both.
- Maximum generations number: $10 \cdot N$.
- Stall generations limit: $8 \cdot N$.

In a first test, we have compared the performance of the algorithms on a small set of problems. Three values of N have been taken into account, *i.e.*, $N = 4, 5, 6$. We have considered two problems with $N = 4, 5$, and one problem with $N = 6$. For these problems, we provide the exact values that parameters take in each of them. This is to favour future comparisons the reader may be interested to accomplish.

Table 2 shows the parameter values, which have been randomly drawn from the intervals in Table 1. Table 3 and Table 4 give the results. Let us begin with Algorithm 3. Results show that it has achieved the best performance, in terms of computational time. Also note that, with respect to Algorithm 1, the computational time reduction is more than 99%. Moreover, the maximum Absolute Percentage Error (APE) does not exceed 0.7%. For what concerns Algorithm 2, it has always reached the same solution as Algorithm 1. Moreover, with respect to Algorithm 1, its computational time reduction is over 99% in higher dimension problems. Algorithm 1 is computationally onerous with respect to both Algorithm 2 and Algorithm 3, however it seems faster than hybrid GA. Algorithm 1 and Algorithm 2 have found the best solutions. In terms of cost efficiency, hybrid GA has performed similarly to Algorithm 1. It has found a worse solution in problem P2 only.

Problem	A_0	Item - n	a_n	h_n	D_n	σ_n	π_n	ρ_n	β_n	$s_{1,n}$ (days)	$s_{2,n}$ (days)	$s_{3,n}$ (days)	$b_{1,n}$ (days)	$b_{2,n}$ (days)	$b_{3,n}$ (days)	$c_{1,n}$ ($/day)	$c_{2,n}$ ($/day)	$c_{3,n}$ ($/day)
P1	172	1	179	18	658	84	81	37	0.25	21	22	18	7	12	7	0.9	2.9	5.1
		2	136	24	693	54	94	63	0.17	20	17	18	13	15	15	0.6	2.2	5.7
		3	145	17	177	20	140	41	0.32	17	18	22	14	12	10	1.0	1.8	4.8
		4	117	12	186	47	81	35	0.64	22	24	19	7	10	13	0.4	2.5	5.6
P2	125	1	181	25	691	199	114	54	0.77	18	18	23	11	8	11	0.8	2.4	5.8
		2	105	12	438	168	99	64	0.66	23	23	22	11	9	11	1.0	2.0	5.1
		3	205	10	443	66	103	62	0.81	17	17	18	11	7	9	0.9	3.1	5.1
		4	205	9	255	48	99	41	0.60	25	23	25	8	14	11	0.4	2.6	4.6
P3	161	1	150	3	551	23	114	29	0.22	21	20	22	11	13	7	0.7	2.9	6.0
		2	112	17	993	232	149	41	0.23	21	23	22	12	12	10	0.6	3.0	5.1
		3	170	14	933	161	110	46	0.15	21	22	23	7	13	8	0.2	2.4	5.0
		4	198	3	608	102	98	43	0.53	18	17	18	11	14	8	0.5	2.3	5.2
		5	163	2	255	5	142	44	0.39	22	23	19	13	10	7	0.7	3.0	4.5
P4	130	1	115	6	607	143	117	52	0.57	23	17	22	15	10	11	0.3	2.8	5.2
		2	183	13	630	10	124	28	0.27	22	18	20	9	12	10	0.9	2.5	4.7
		3	148	2	213	27	144	53	0.45	18	24	24	13	8	12	0.5	1.9	4.4
		4	178	17	913	327	107	61	0.16	17	20	21	12	12	8	0.2	3.0	4.8
		5	135	11	932	307	96	51	0.71	19	23	21	9	9	14	0.9	2.8	5.0
P5	177	1	155	21	195	36	84	70	0.28	23	21	21	15	10	12	0.8	2.2	5.3
		2	106	4	612	128	129	28	0.65	19	18	24	13	14	9	0.6	2.0	5.3
		3	142	1	807	42	143	37	0.76	18	21	22	10	8	14	0.5	2.6	5.0
		4	211	19	994	387	128	45	0.59	24	22	18	15	9	8	0.7	2.6	4.6
		5	196	4	995	221	87	30	0.26	25	20	23	7	14	10	0.6	2.5	5.8
		6	141	19	886	105	148	44	0.44	19	18	18	10	7	8	0.3	2.8	4.2

Table 2. Parameter values adopted in the first test.

		Algorithm 1						Hybrid Genetic Algorithm					
Prob.	**Item**	k_n	*A*	L_n	*T*	*Cost*	*Comp. time (sec)*	k_n	*A*	L_n	*T*	*Cost*	*Comp. time (sec)*
P1	1	1		26				1		26			
	2	1		43				1		43			
	3	2		36				2		36			
	4	2		30				2		30			
			120.5		0.21	15343	32.5		120.5		0.21	15343	986
P2	1	1		30				1		30			
	2	1		31				1		31			
	3	1		27				1		27			
	4	2		33				1		33			
			125		0.31	32215	34.4		125		0.34	32253	1649
P3	1	2		31				2		31			
	2	1		34				1		34			
	3	1		28				1		28			
	4	2		33				2		33			
	5	3		30				3		30			
			149.5		0.26	24522	1723		149.5		0.26	24522	9514
P4	1	1		36				1		36			
	2	1		31				1		31			
	3	3		33				3		33			
	4	1		32				1		32			
	5	1		32				1		32			
			130		0.32	39285	2257		130		0.32	39285	9764
P5	1	2		37				2		37			
	2	2		36				2		36			
	3	3		32				3		32			
	4	2		32				2		32			
	5	2		31				2		31			
	6	1		25				1		25			
			108.6		0.19	45669	33686		108.6		0.19	45669	46090

Table 3. Results of the first test. Data relevant to Algorithm 1 and hybrid GA.

		Algorithm 2						Algorithm 3					
Prob.	**Item**	k_n	*A*	L_n	*T*	*Cost*	*Comp. time (sec)*	k_n	*A*	L_n	*T*	*Cost*	*Comp. time (sec)*
P1	1	1		26				1		26			
	2	1		43				1		43			
	3	2		36				2		36			
	4	2		30				2		30			
			120.5		0.21	15343	3.16		120.0		0.21	15343	0.17
P2	1	1		30				1		30			
	2	1		31				1		31			
	3	1		27				2		27			
	4	2		33				2		33			
			125		0.31	32215	5.11		125		0.25	32439	0.24
P3	1	2		31				2		31			
	2	1		34				1		34			
	3	1		28				1		28			
	4	2		33				3		33			
	5	3		30				4		30			
			149.5		0.26	24522	19.74		125.7		0.22	24627	0.79
P4	1	1		36				1		36			
	2	1		31				1		31			
	3	3		33				4		33			
	4	1		32				1		32			
	5	1		32				1		32			
			130		0.32	39285	21.17		130		0.26	39505	1.09
P5	1	2		37				2		37			
	2	2		36				2		36			
	3	3		32				3		32			
	4	2		32				1		32			
	5	2		31				2		31			
	6	1		25				1		25			
			108.6		0.19	45669	66.08		130.4		0.22	45962	3.75

Table 4. Results of the first test. Data relevant to Algorithm 2 and Algorithm 3.

A second run of experiments has successively been carried out to investigate the performance of the algorithms on a bigger set of (random) problems, taking also into account a larger number of items. Table 5 shows the results. Note that, for problems with $N > 6$, Algorithm 1 and hybrid GA have not been executed. In fact, their computational time has turned out to be unpractical. The observations that we have previously made can substantially be confirmed here. In addition, we can note that Algorithm 2 has been able to reach the same solution of Algorithm 1 in over 80% of cases. In the others, the APE has typically turned out to be smaller than 0.1% (the maximum value is 0.3%). Moreover, contrarily to the other algorithms, the computational time of Algorithm 1 appears remarkably variable, among problems with same dimension. It is significant that the APE of Algorithm 3 has reached values not greater than 0.8% (for problems with $N > 6$, it has been evaluated with respect to Algorithm 2).

No. of items	Computational time (sec.)				Cost of solution			
	Alg. 1	*Alg. 2*	*Alg. 3*	*Hybrid GA*	*Alg. 1*	*Alg. 2*	*Alg. 3*	*Hybrid GA*
4	100	5.0	0.25	1689	**22696**	22750 (0.2%)	22725 (0.1%)	**22696**
	34	3.8	0.22	1042	**13575**	**13575**	13618 (0.3%)	**13575**
	9	4.5	0.25	1379	**18296**	**18296**	**18296**	**18296**
	193	4.7	0.17	1515	**21689**	**21689**	22155 (2.1%)	**21689**
	46	3.2	0.17	1051	**17266**	17275 (< 0.1%)	17275 (< 0.1%)	**17266**
	8	3.2	0.17	1018	**31900**	**31900**	31923 (< 0.1%)	**31900**
	45	4.7	0.25	1532	**23147**	**23147**	23148 (< 0.1%)	**23147**
	158	4.6	0.24	1446	**33150**	33160 (< 0.1%)	33350 (0.6%)	**33150**
	49	3.1	0.19	1034	**14992**	**14992**	14999 (< 0.1%)	**14992**
	52	4.6	0.23	1493	**11176**	**11176**	**11176**	**11176**
5	63	14.5	0.80	6565	**28883**	**28883**	29099 (0.7%)	**28883**
	368	22.4	1.10	10711	**28580**	28665 (0.3%)	28676 (0.3%)	**28580**
	948	14.3	0.80	6392	**24717**	24754 (0.1%)	24873 (0.6%)	**24717**
	4574	22.2	0.80	10901	**34788**	**34788**	35258 (1.4%)	**34788**
	593	20.9	1.08	10104	**18424**	18435 (< 0.1%)	18427 (< 0.1%)	**18424**
	662	22.2	1.06	10286	**32225**	32330 (0.3%)	32341 (0.4%)	**32225**
	1941	21.1	1.08	10389	**26521**	**26521**	26540 (< 0.1%)	**26521**
	3184	21.3	1.09	10269	**15737**	15750 (< 0.1%)	15750 (< 0.1%)	**15737**
	1385	15.3	0.84	7818	**29082**	29087 (< 0.1%)	29103 (< 0.1%)	**29082**
	5263	21.1	1.06	10236	**25590**	**25590**	25598 (< 0.1%)	**25590**
6	8192	91.9	4.70	65536	**19235**	**19235**	19238 (< 0.1%)	**19235**
	9625	93.8	4.67	57344	**40561**	**40561**	40649 (0.2%)	**40561**
	307200	93.6	3.53	65480	**45294**	**45294**	45673 (0.8%)	**45294**
	102400	90.9	3.50	61425	**36066**	**36066**	36594 (1.5%)	**36066**
	8545	95.1	3.56	57490	**38240**	**38240**	38371 (0.3%)	**38240**
	4212	72.4	3.84	45056	**19719**	**19719**	19725 (< 0.1%)	**19719**
	4343	94.7	4.50	58652	**36592**	36595 (< 0.1%)	36630 (0.1%)	36595 (< 0.1%)
	28672	66.8	3.52	47810	**23584**	23587 (< 0.1%)	23589 (< 0.1%)	**23584**
	4152	65.9	3.52	46742	**38032**	**38032**	38042 (< 0.1%)	**38032**
	4796	93.3	4.66	61440	**31087**	**31087**	31126 (0.1%)	**31087**
7	-	406.3	19.11	-	-	**47768**	48155 (0.8%)	-
	-	414.2	20.20	-	-	**42196**	42405 (0.5%)	-
	-	446.9	20.24	-	-	**36071**	36207 (0.4%)	-
	-	407.8	15.61	-	-	**34055**	34078 (< 0.1%)	-
	-	340.2	15.58	-	-	**55275**	55424 (0.3%)	-
	-	342.0	19.35	-	-	**33209**	33210 (< 0.1%)	-
	-	299.7	16.02	-	-	**25046**	25161 (0.5%)	-
	-	301.3	15.56	-	-	**44560**	44638 (0.2%)	-
	-	395.0	15.59	-	-	**48460**	48555 (0.2%)	-
	-	414.8	17.35	-	-	**36824**	37066 (0.7%)	-
8	-	1810	68.39	-	-	**63344**	**64344**	-
	-	1364	67.77	-	-	**39129**	39133 (< 0.1%)	-
	-	1643	84.95	-	-	**31179**	31205 (< 0.1%)	-
	-	1406	71.54	-	-	**47049**	47064 (< 0.1%)	-
	-	1290	69.04	-	-	**33373**	33410 (0.1%)	-
	-	1586	68.21	-	-	**35868**	35891 (< 0.1%)	-
	-	1319	68.16	-	-	**49653**	49761 (0.2%)	-
	-	1305	78.55	-	-	**40815**	40847 (< 0.1%)	-
	-	1781	87.88	-	-	**38133**	38412 (0.7%)	-
	-	1317	68.02	-	-	**69793**	70103 (0.4%)	-

Table 5. Results of extended tests. Bold values are minimum costs. Percentage within brackets represents the APE.

With regard to the comparative analysis between solution methods, a final set of tests has been executed to investigate the performance of the algorithms on (random) problems with even greater dimension and in a more circumscribed setting. That is, in these new experiments, we have neglected the possibility to control lead times and major ordering cost in order to put our attention on a more focused context. Therefore, problems have concerned the optimization of the replenishment policy only. Table 6 gives the results. Note that Algorithm 1 has not been considered since the computational time turned out to be excessively high. In each problem, Algorithm 3 has required less than 0.01 seconds, while Algorithm 2 has run for a time in the order of a few centiseconds. The computational time of hybrid GA is significantly higher. We can further observe that Algorithm 2 has been the most efficient algorithm in about 70% of cases, while in the others the mean APE is about 0.2%. The solutions that Algorithm 3 has found are substantially efficient: the maximum APE is near 1.9%. Finally, we would observe that hybrid GA has been the most efficient algorithm in about 50% of cases.

The last tests have concerned the comparison of our distribution-free model with respect to the case in which the demand distribution is known. That is, the objective has consisted in evaluating the error our model achieves with respect to the same inventory model in the circumstance where the demand distribution is completely defined (with the same values of D_n and σ_n in both models). In particular, as "known" model, we have considered the case in which the demand in the protection interval is Gaussian. Note that we have taken into account the Gaussian distribution as this approach is widely spread in practice as well as in literature (Moon and Gallego [25]; Andersson *et al*. [26]; Gallego *et al*. [27]). As error measure we have adopted the following expression:

$$E_{\%} \equiv \frac{C_N\left(\mathbf{q}^*\right) - C_N\left(\mathbf{q}_N^*\right)}{C_N\left(\mathbf{q}_N^*\right)} \times 100 \,, \tag{33}$$

where C_N is the expected total cost function for the case with Gaussian demand in the protection interval, $\mathbf{q}_N^*$ is the minimum-cost vector for the model with Gaussian demand, and $\mathbf{q}^*$ is the minimum-cost vector for the distribution-free model. Clearly, $\mathbf{q}_N^*$ and $\mathbf{q}^*$ are referred to a same problem.

This comparison has been carried out on several random problems, taking into reference the more focused condition in which lead times and major ordering cost are fixed, rather than decision variables. This has permitted us to investigate cases with a larger number of items. Each problem has been generated by randomly drawing parameter values from the intervals shown in Table 1. Since it has shown a very good trade-off in terms of cost of solution and computational efficiency, Algorithm 2 has been adopted to find $\mathbf{q}_N^*$ and $\mathbf{q}^*$. The results of this analysis are given in Table 7. We can first note that the error is increasing in *N*. For what concerns the magnitude, while it is smaller than 10% for $N < 40$, for $N \geq 40$ the error is slightly larger than 10%: the greatest value is about 11.3%. In conclusion, if we also consider that, in similar comparisons, Moon and Gallego [25] achieved a maximum percentage error near 33%, we may substantially judge as positive the performance of our distribution-free model with respect to the case in which the demand in the protection interval is Gaussian.

No. of items	Computational time (sec.)			Cost of solution		
	Algorithm 2	*Algorithm 3*	*Hybrid GA*	*Algorithm 2*	*Algorithm 3*	*Hybrid GA*
10	0.02	< 0.01	33.0	**59695**	60467 (1.3%)	**59695**
	0.03	< 0.01	32.1	**45932**	45986 (0.1%)	**45932**
	0.03	< 0.01	31.5	**72574**	73427 (1.2%)	**72574**
11	0.03	< 0.01	39.8	**83411**	84180 (0.9%)	**83411**
	0.03	< 0.01	38.9	**101171**	103053 (1.9%)	**101171**
	0.03	< 0.01	39.2	86513 (< 0.1%)	87592 (1.3%)	**86490**
12	0.02	< 0.01	49.0	46994 (0.2%)	46952 (< 0.1%)	**46919**
	0.03	< 0.01	49.7	66187 (0.6%)	66196 (0.6%)	**65770**
	0.03	< 0.01	51.8	54882 (0.3%)	54873 (0.3%)	**54711**
13	0.03	< 0.01	54.8	**80714**	82126 (1.7%)	**80714**
	0.03	< 0.01	61.6	**76001**	76438 (0.6%)	**76001**
	0.03	< 0.01	57.2	67402 (< 0.1%)	67565 (0.3%)	**67341**
14	0.03	< 0.01	70.9	**99803**	100402 (0.6%)	100081 (0.3%)
	0.03	< 0.01	67.5	**57732**	**57732**	**57732**
	0.03	< 0.01	65.0	104123 (0.1%)	104434 (0.4%)	**104017**
15	0.03	< 0.01	89.0	126793 (< 0.1%)	127720 (0.8%)	**126715**
	0.03	< 0.01	82.4	**96137**	96742 (0.6%)	**96137**
	0.05	< 0.01	87.3	**103921**	104248 (0.3%)	104199 (0.3%)
16	0.05	< 0.01	93.1	**110699**	110910 (0.2%)	**110699**
	0.03	< 0.01	102.0	73107 (0.2%)	73587 (0.8%)	**72980**
	0.03	< 0.01	98.5	**95300**	96110 (0.8%)	95376 (< 0.1%)
17	0.03	< 0.01	112.0	**80765**	80807 (< 0.1%)	81944 (1.5%)
	0.05	< 0.01	112.9	**94542**	94619 (< 0.1%)	94873 (0.4%)
	0.05	< 0.01	123.3	**100601**	100750 (0.1%)	100623 (< 0.1%)
18	0.05	< 0.01	136.1	103109 (< 0.1%)	103738 (0.7%)	**103056**
	0.05	< 0.01	132.1	**103132**	103782 (0.6%)	103401 (0.3%)
	0.05	< 0.01	129.3	136695 (0.4%)	136921 (0.6%)	**136172**
19	0.05	< 0.01	145.6	**124420**	125622 (1.0%)	124705 (0.2%)
	0.05	< 0.01	154.6	**1.0315**	1.0326 (0.1%)	1.0316 (< 0.1%)
	0.05	< 0.01	150.6	1.0542 (0.2%)	1.0579 (0.5%)	**1.0522**
20	0.05	< 0.01	168.5	**0.9747**	0.9801 (0.6%)	0.9756 (< 0.1%)
	0.05	< 0.01	172.4	0.9482 (0.1%)	0.9494 (0.3%)	**0.9470**
	0.06	< 0.01	174.8	**1.5754**	1.5831 (0.5%)	1.5767 (< 0.1%)
21	0.05	< 0.01	184.2	**1.3743**	1.3769 (0.2%)	1.3816 (0.5%)
	0.06	< 0.01	186.7	**1.2142**	1.2169 (0.2%)	1.2168 (0.2%)
	0.06	< 0.01	187.9	1.5987 (< 0.1%)	1.6040 (0.4%)	**1.5975**
22	0.05	< 0.01	220.2	**1.5896**	1.6047 (0.9%)	1.5957 (0.4%)
	0.06	< 0.01	207.3	**1.3194**	1.3290 (0.7%)	1.3197 (< 0.1%)
	0.06	< 0.01	209.5	**1.6254**	1.6294 (0.2%)	1.6271 (0.1%)
23	0.06	< 0.01	229.4	**1.3647**	1.3677 (0.2%)	1.3693 (0.3%)
	0.05	< 0.01	229.5	1.8991 (0.1%)	1.9051 (0.4%)	**1.8969**
	0.06	< 0.01	246.6	**1.0211**	1.0217 (< 0.1%)	1.0243 (0.3%)
24	0.06	< 0.01	267.1	**1.2107**	1.2160 (0.4%)	1.2126 (0.2%)
	0.06	< 0.01	252.3	**1.0522**	1.0539 (0.2%)	1.0532 (< 0.1%)
	0.06	< 0.01	270.8	**1.3896**	1.3941 (0.3%)	1.3955 (0.4%)
25	0.08	< 0.01	304.6	**1.1024**	1.1075 (0.5%)	1.1028 (< 0.1%)
	0.06	< 0.01	318.1	**1.5538**	1.5589 (0.3%)	1.5645 (0.7%)
	0.06	< 0.01	297.6	**1.6319**	1.6436 (0.7%)	**1.6319**

Table 6. Results of tests in a more focused context. Bold values are minimum costs. Percentage within brackets represents the APE.

No. of items	$E_{\%}$	No. of items	$E_{\%}$
	7.2%		9.9%
	5.1%		8.9%
	8.4%		7.3%
4	4.9%	*32*	9.4%
	8.2%		9.8%
	5.8%		8.3%
	5.2%		9.6%
	8.6%		8.8%
	2.9%		9.8%
	5.0%		9.9%
8	8.5%	*36*	7.8%
	8.3%		9.7%
	8.2%		9.6%
	6.7%		9.6%
	9.4%		6.7%
	9.7%		9.6%
	7.8%		9.3%
12	9.1%	*40*	7.1%
	8.2%		10.0%
	8.5%		10.8%
	4.4%		6.9%
	6.8%		11.3%
	6.6%		7.7%
	6.6%		10.4%
16	8.6%	*50*	7.8%
	7.0%		10.5%
	8.6%		11.2%
	7.2%		9.7%
	7.6%		10.6%
	7.5%		9.9%
	7.6%		11.3%
20	6.9%	*60*	10.9%
	8.4%		9.8%
	7.3%		10.3%
	9.6%		11.2%
	8.5%		9.7%
	8.5%		9.6%
	8.7%		10.1%
	8.7%		11.1%
24	8.3%	*70*	9.5%
	7.0%		10.6%
	9.6%		10.8%
	8.5%		11.5%
	6.9%		11.1%
	7.2%		11.3%
	9.5%		9.8%
28	9.4%	*80*	9.8%
	7.1%		10.7%
	9.2%		10.4%
	8.2%		10.2%

Table 7. Results of the comparison between models with distribution-free procedure and Gaussian demand.

6. Conclusions and further remarks

In this paper, we studied the periodic-review stochastic JRP with backorders-lost sales mixtures. We assumed that lead times are made of two major components: a common part to all items and an item-specific portion. The item-specific component of lead times was supposed to be controllable, as well as the major ordering cost. The expected total cost function was developed exploiting the minimax distribution-free approach. The objective was to find the strict cyclic replenishment policy, the length of (the item-specific component of) lead times, and the major ordering cost that minimize the long-run expected total cost.

To solve this problem, we presented a first optimization algorithm (Algorithm 1) and two efficient alternative heuristics (Algorithm 2 and Algorithm 3). These alternative solution procedures work similarly to a standard algorithm suitable for the deterministic JRP. They however differ in the fact that one of them works on an approximated expression of the cost function. This approximation was obtained by replacing part of the cost function with an appropriate Taylor series expansion.

Extensive numerical experiments were then carried out to investigate the performance of the developed algorithms. Algorithm 1 turned out to be computationally onerous, to such an extent that, for problems with seven or more items, it may be unpractical. Algorithm 2 and Algorithm 3 demonstrated to be highly efficient in terms of both cost of solution and computational time; they seem therefore promising for a practical application.

We finally compared, under a numerical viewpoint, our distribution-free model with respect to the same inventory model in the case in which the demand in the protection interval is Gaussian. Results showed that the percentage error, under the distribution-free approach, is increasing with N. However, even in the case with $N = 80$, it was always below 12%.

A remark concerning the common component L of lead times is needed. In our model, we assumed that it is deterministic and constant. However, the extension to the case in which L is controllable is relatively immediate. In this regard, it is possible to adopt, for example, the following approach. The crashing cost of L can be treated similarly to the major ordering cost (or embedded into the major ordering cost). Hence, the cases with different L can be represented by different scenarios. Each scenario can be tackled in a way with controllable L_n, as we have already done in our model, and fixed L. The optimal L can then be found by identifying the best scenario *i.e*., the one that achieves the minimum expected total cost.

Future researches may be devoted to further extensions of the JRP, for example in an integrated vendor-buyer supply chain. In addition, it may also be possible to deepen the analysis about the sensitivity of the error achieved by the proposed heuristics with respect to model parameters.

Another plausible extension may be considering a different distribution-free method to evaluate the long-run expected total cost. It is, in fact, recognized that the *minimax* procedure disseminated by Moon and Gallego [25] is particularly conservative and characterizes an extremely risk adverse firm. Therefore, it may be interesting proposing a different distribution-free JRP model, with the further objective of comparing the results obtained in this paper.

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